On The Moment Problem

Chia-Linn Wu

Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan, 804
R. O. C.

June, 2002
Contents

1 Introduction 1

2 Conditions for moment-determinacy 4

3 A special case for the discrete distribution function 9

References 13
Abstract

Let $F$ be a distribution function and $\{m_n\}_{n=1}^{\infty}$ be its moments. The moment problem is to know whether the moments $\{m_n\}_{n=1}^{\infty}$ determine the distribution function $F$. In general, the sequence of moments does not always determine the distribution function. So the conditions for a distribution function to be moment-determinate are investigated. We get a result concerning the discrete distribution function.
1 Introduction

A real-valued function $F$ with domain $(-\infty, \infty)$ which satisfies the following three conditions:

(i) non-decreasing,

(ii) right-continuous,

(iii) $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$

is called a distribution function.

Let $F$ be a distribution function and $\{m_n\}_{n=1}^{\infty}$ be its moments of all orders, i.e.,

$$m_n = \int_{-\infty}^{\infty} x^n dF(x).$$

The moment problem is to know whether the moments $\{m_n\}_{n=1}^{\infty}$ determine the distribution function $F$. More precisely, let $F$ and $G$ be distribution functions with the same moments, i.e.,

$$\int_{-\infty}^{\infty} x^n dF(x) = \int_{-\infty}^{\infty} x^n dG(x)$$

for all integers $n \geq 1$. The moment problem is to know whether $F$ and $G$ must be the same. If $F$ and $G$ are the same, then we say that the moments $\{m_n\}_{n=1}^{\infty}$ determine the distribution function $F$ (i.e., there exists no other distribution function with the same moments as $F$).

The motivation to investigate the moment problem is the following theorem, which is often referred to as the method of moments.

**Theorem ([3, p.99]).** Suppose there is a unique distribution function $F$ with the moments $\{m_n\}_{n=1}^{\infty}$, all finite. Suppose that $\{F_k\}$ is a sequence of distribution functions, each of which has all its moments finite:

$$m_n^{(k)} = \int_{-\infty}^{\infty} x^n dF_k.$$

Finally, suppose that for every $n \geq 1$:

$$\lim_{k \to \infty} m_n^{(k)} = m_n.$$
Then $F_k \xrightarrow{u} F$, that is, $\lim_{k \to \infty} F_k(x) = F(x)$ for all continuity points $x$ of $F$.

Next we want to ask the following question: Do the moments $\{m_n\}_{n=1}^\infty$ always determine the distribution function?

In general, the answer is “No.” To see this, we give an example of a distribution function which is not determined uniquely by its moments.

**Example** ([1, p.294]). Consider the distribution function $F$ with density

$$f(x) = \begin{cases} 
ke^{-\alpha x^\lambda}, & x > 0, \\
0, & x \leq 0,
\end{cases}$$

where $\alpha > 0, 0 < \lambda < \frac{1}{2}$, and $k$ is determined by the condition $\int_0^\infty f(x)dx = 1$.

Let

$$g(x) = \begin{cases} 
ke^{-\alpha x^\lambda}[1 + \epsilon \sin(\beta x^\lambda)], & x > 0, \\
0, & x \leq 0,
\end{cases}$$

where $\beta = \alpha \tan \lambda \pi$ and $|\epsilon| < 1$.

Then $g(x) \geq 0$ and

$$\int_0^\infty x^n e^{-\alpha x^\lambda} \sin \beta x^\lambda dx = 0 \quad (1)$$

for all integers $n \geq 0$.

Let $G(x)$ be the distribution function with density $g(x)$. It follows from (1) that the distribution functions $F$ and $G$ have equal moments, i.e.,

$$\int_{-\infty}^\infty x^n dF(x) = \int_{-\infty}^\infty x^n dG(x)$$

for all integers $n \geq 0$, but $F$ is different from $G$.

So our goal is to give some conditions that guarantee the uniqueness of the solution of the moment problem.
Let $F$ be a distribution function with finite moments of all orders. We say that the distribution function $F$ is moment-determinate if it is uniquely determined by its moments $\{m_n\}_{n=1}^{\infty}$; otherwise, $F$ is moment-indeterminate.

In section 2, we will present some sufficient conditions for a distribution function to be moment-determinate. Besides, we will also present sufficient conditions for a distribution function to be moment-indeterminate (Theorem 2.4 and Theorem 2.6). In section 3, we will obtain a new result about the distribution function to be moment-determinate. Moreover, we are interested in the case of the discrete distribution function supported on $\mathbb{N}$ and get a result for the moment problem.
2 Conditions for moment-determinacy

It is well known that if the distribution function $F$ is concentrated on a finite interval, then $F$ is moment-determinate ([1, p.295]).

But the corresponding result with the distribution function supported on an infinite interval is false. By giving some extra information about the moment sequence, we have the following result.

**Theorem 2.1 ([1, p.295]).** Let $F = F(x)$ be a distribution function and $\mu_n = \int_{-\infty}^{\infty} |x|^n dF(x)$. If
$$\lim_{n \to \infty} \frac{\mu_n^{1/n}}{n} < \infty,$$
then the distribution function $F = F(x)$ is moment-determinate.

From Theorem 2.1, we also have the following result.

**Theorem 2.2 ([1, p.296]).** Let $F = F(x)$ be a distribution function. If
$$\lim_{n \to \infty} \frac{(m_{2n})^{1/2n}}{2n} < \infty,$$
then the distribution function $F = F(x)$ is moment-determinate.

Next we present the Carleman’s test for the uniqueness of the moment problem.

**Theorem 2.3 ([1, p.296]).** Let $\{m_n\}_{n=1}^\infty$ be the moments of a distribution function $F$. If
$$\sum_{n=0}^{\infty} \frac{1}{(m_{2n})^{1/2n}} = \infty,$$
then the distribution function $F$ is moment-determinate.
Finally we present some results concerning the distribution function with positive density function. The following two results are concerned with distributions supported on the whole line $\mathbb{R} \equiv (-\infty, \infty)$.

**Theorem 2.4 ([2]).** Let $F$ be an absolutely continuous distribution function with density function $f(x) > 0$ for all $x \in \mathbb{R}$, and let $F$ have finite moments of all orders. Assume further that the Lebesgue integral

$$\int_{-\infty}^{\infty} \frac{-\log f(x)}{1 + x^2} dx < \infty.$$  \hspace{1cm} (2)

Then the distribution function $F$ is moment-indeterminate.

We shall apply the results of Hardy theory to prove Theorem 2.4. First we give a definition of Hardy space. The Hardy space $H^1$ is the set of all analytic functions on the upper half-plane such that

$$\sup_{0 < y < \infty} \int_{-\infty}^{\infty} |f(x + iy)| dx < \infty.$$  \hspace{1cm} (3)

The following lemmas are two results of $H^1$ theory.

**Lemma 1.** Let $h > 0$ be a Lebesgue integrable function on $\mathbb{R}$. Assume that the integral

$$\int_{-\infty}^{\infty} \frac{-\log h(x)}{1 + x^2} dx < \infty.$$  \hspace{1cm} (4)

Then there exists a $g \in H^1$ such that $|g(x)| = h(x)$ almost everywhere (a.e.) on $\mathbb{R}$.

**Lemma 2.** Let $g \in H^1$. Then the Fourier transform

$$\int_{-\infty}^{\infty} g(x)e^{itx} dx = 0$$

for all $t \geq 0$.

**Proof of Theorem 2.4.** By condition (2) and Lemma 1, there exists a function $g \in H^1$ such that $|g(x)| = f(x)$ a.e. on $\mathbb{R}$. Further, from Lemma 2 it follows that

$$\int_{-\infty}^{\infty} g(x)e^{itx} dx = 0$$  \hspace{1cm} (3)

for all \( t \geq 0 \).

Since the distribution \( F \) has finite moments of all orders, we can make repeated differentiations on both sides of relation (3) and obtain that

\[
\int_{-\infty}^{\infty} (ix)^k g(x)e^{itx}dx = 0
\]

for \( t \geq 0 \) and for \( k = 0, 1, 2, ... \)

Putting \( t = 0 \) in the last relation yields

\[
\int_{-\infty}^{\infty} x^k g(x)dx = 0
\]

for \( k = 0, 1, 2, ... \), so that

\[
\int_{-\infty}^{\infty} x^k \Re g(x)dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} x^k \Im g(x)dx = 0
\]

for \( k = 0, 1, 2, ... \), where \( \Re g \) and \( \Im g \) denote the real and imaginary parts of \( g \), respectively.

Since \( |g(x)| > 0 \) a.e. on \( \mathbb{R} \), one of \( \Re g \) and \( \Im g \) is a nonzero function, say the real part \( \Re g \). Define the real-valued function

\[
h(x) = f(x) + \Re g(x)
\]

for \( x \in \mathbb{R} \).

Then \( h \geq 0 \), \( h \not= f \) and

\[
\int_{-\infty}^{\infty} x^k h(x)dx = \int_{-\infty}^{\infty} x^k f(x)dx \quad \text{for } k = 0, 1, 2, ...
\]

Namely, the density function \( h \) is different from \( f \), but it has exactly the same moments of \( f \). Therefore, the distribution function \( F \) is moment-indeterminate.

\[ \square \]

**Theorem 2.5 ([2]).** Let \( F \) be an absolutely continuous distribution function with density function \( f(x) > 0 \) for all \( x \in \mathbb{R} \). Let \( F \) have finite moments of all orders, and
let $f$ be symmetric about zero and differentiable on $\mathbb{R}$. Suppose there exists an $x_0 \geq 0$ such that $f(x)$ decreases to zero and $-\frac{xf'(x)}{f(x)}$ increases to infinity as $x_0 < x \to \infty$. Assume further that the integral
\[
\int_{-\infty}^{\infty} \frac{-\log f(x)}{1 + x^2} dx = \infty.
\]
Then $F$ is moment-determinate.

**Proof.** Write
\[
(x^{2n}f(x))' = x^{2n-1}f(x)[2n + xf'(x)/f(x)] \text{ for } x > 0.
\]
Then by the assumption $-xf'(x)/f(x) \uparrow \infty$ as $x_0 < x \to \infty$, there exist a positive integer $m$ and a sequence of real numbers $x_0 \leq x_m < x_{m+1} < \ldots < x_n \uparrow \infty$ such that
\[
x_{2n}f(x_n) = \sup_{x \geq x_0} x^{2n}f(x) \text{ for } n \geq m.
\]
Now we estimate the moments from above:
\[
\mu_{2n} \equiv 2 \int_{0}^{\infty} x^{2n} f(x) dx
\]
\[
= 2 \left[ \int_{0}^{x_m} x^{2n} f(x) dx + \int_{x_m}^{\infty} \frac{x^{2n+2} f(x)}{x^2} dx \right]
\]
\[
\leq C x_{n+1}^{2n+2}
\]
for $n \geq m$, where $C$ is a positive constant.

This means that if $\sum_{n=m}^{\infty} x_n^{-1} = \infty$, then $\sum_{n=m}^{\infty} \mu_{2n}^{-1/(2n)} = \infty$ and hence $F$ is moment-determinate. To estimate the first sum from below, we apply integration by parts and obtain that for $n > m$
\[
- \int_{x_m}^{x_n} \frac{1}{x^2} \log f(x) dx - \frac{1}{x_n} \log f(x_n) + \frac{1}{x_m} \log f(x_m)
\]
\[
= \int_{x_m}^{x_n} \frac{1}{x^2} \left( -\frac{xf'(x)}{f(x)} \right) dx \leq \sum_{p=m+1}^{n} \int_{x_{p-1}}^{x_p} \frac{2p}{x^2} dx
\]
\[
= \sum_{p=m+1}^{n} 2p \left( \frac{1}{x_{p-1}} - \frac{1}{x_p} \right) \leq (2m + 2) \sum_{p=m}^{n} \frac{1}{x_p},
\]
in which the first inequality is due to the fact that $x^{2p}f(x)$ is increasing for $x \leq x_p$, making the quantity in (5) (with $n$ replaced by $p$) nonnegative. Letting $n \to \infty$ together with the condition (4) yields $\sum_{p=m}^{\infty} x_p^{-1} = \infty$.

□

The next two results are concerned with distributions supported on the half-line $\mathbb{R}_+ \equiv (0, \infty)$. For the proof of Theorem 2.6, it needs to construct a distribution function $G$ relating to the distribution function $F$ such that the density function of $G$ is positive in $\mathbb{R}$, and then use Theorem 2.4. As for Theorem 2.7, we can prove it by using Carleman’s condition as in the proof of Theorem 2.5.

**Theorem 2.6 ([2]).** Let $F$ be an absolutely continuous distribution function with density function $f(x) > 0$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. Assume that $F$ has finite moments of all orders and that the integral
\[
\int_0^\infty -\log f(x^2) \frac{1}{1 + x^2} \, dx < \infty.
\]
Then the distribution function $F$ is moment-indeterminate.

**Theorem 2.7 ([2]).** Let $F$ be an absolutely continuous distribution function with density function $f(x) > 0$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. Let $F$ have finite moments of all orders and let $f$ be differentiable on $\mathbb{R}_+$. Suppose there exists an $x_0 \geq 0$ such that $f(x)$ decreases to zero and $-\frac{x f'(x)}{f(x)}$ increases to infinity as $x_0 < x \to \infty$. Assume further that the integral
\[
\int_0^\infty -\log f(x^2) \frac{1}{1 + x^2} \, dx = \infty.
\]
Then $F$ is moment-determinate.
3 A special case for the discrete distribution function

First we give a result about the moment problem.

**Theorem 3.1 ([4, p.388]).** Let $F$ be a distribution function having finite moments $m_k = \int_{-\infty}^{\infty} x^k dF(x)$ of all orders. If the power series $\sum_k t^k m_k / k!$ has a positive radius of convergence, then the distribution function $F$ is moment-determinate.

**Proof.** Let $\alpha_k = \int_{-\infty}^{\infty} |x|^k dF(x)$ be the absolute moments. The first step is to show that

$$\frac{t^k \alpha_k}{k!} \to 0, \quad k \to \infty,$$

for some positive $t$. By hypothesis there exists an $s$, $0 < s < 1$, such that $s^k m_k / k! \to 0$. Choose $0 < t < s$; then $2kt^{2k-1} < s^{2k}$ for large $k$. Since $|x|^{2k-1} \leq 1 + |x|^{2k}$,

$$\frac{t^{2k-1} \alpha_{2k-1}}{(2k-1)!} \leq \frac{t^{2k-1}}{(2k-1)!} + \frac{s^{2k} \alpha_{2k}}{(2k)!}$$

for large $k$. Hence (6) holds as $k$ goes to infinity through odd values; since $\alpha_k = m_k$ for $k$ even, (6) follows.

By $|e^{ix} - \sum_{k=0}^{n} \left(\frac{(ix)^k}{k!}\right)| \leq \min\{\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!}\}$,

$$|e^{irx} \left(e^{ihx} - \sum_{k=0}^{n} \frac{(ihx)^k}{k!}\right)| \leq \left|h x\right|^n \frac{1}{(n+1)!},$$

and therefore the characteristic function $\varphi$ of $F$ satisfies

$$|\varphi(r + h) - \sum_{k=0}^{n} \frac{h^k}{k!} \int_{-\infty}^{\infty} (ix)^k e^{irx} dF(x)| \leq \frac{|h|^{n+1} \alpha_{n+1}}{(n+1)!}.$$

By $\varphi^{(k)}(r) = \int_{-\infty}^{\infty} (ix)^k e^{irx} dF(x)$, the integral here is $\varphi^{(k)}(r)$. By (6),

$$\varphi(r + h) = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(r)}{k!} h^k, \quad |h| \leq t. \quad (7)$$
If \( G \) is another distribution function with moments \( m_k \) and characteristic function \( \psi(r) \), the same argument gives
\[
\psi(r + h) = \sum_{k=0}^{\infty} \frac{\psi^k(r)}{k!} h^k, \quad |h| \leq t.
\] (8)

Take \( r = 0 \); since \( \varphi^k(0) = i^k m_k = \psi^k(0) \), \( \varphi \) and \( \psi \) agree in \((-t, t)\) and hence have identical derivatives there. Taking \( r = t - \epsilon \) and \( r = -t + \epsilon \) in (7) and (8) shows that \( \varphi \) and \( \psi \) also agree in \((-2t + \epsilon, 2t - \epsilon)\) and hence in \((-2t, 2t)\). But then they must by the same argument agree in \((-3t, 3t)\) as well, and so on. Thus \( \varphi \) and \( \psi \) coincide, and by the uniqueness theorem for characteristic functions, so do \( F \) and \( G \).

Next we derive the following result by using Theorem 3.1.

**Theorem 3.2** Let \( F \) be a distribution function having finite moments \( m_n = \int_{-\infty}^{\infty} x^n dF(x) \) of all orders. If
\[
\lim_{n \to \infty} \frac{1}{n!} |m_{n+1} - m_n| = L < \infty,
\]
then the distribution function \( F \) is moment-determinate.

**Proof.** Since
\[
\lim_{n \to \infty} \frac{|t^{n+1}m_{n+1}/(n+1)! - t^n m_n/n!|}{|t^n m_n/n!|} = \lim_{n \to \infty} \frac{|t| \cdot |m_{n+1}/m_n|}{|n+1|} = |t| \cdot \lim_{n \to \infty} \frac{|m_{n+1}/m_n|}{n+1} = |t| \cdot L.
\]

If \( |t| < \frac{1}{L} \), then
\[
\lim_{n \to \infty} \frac{|t^{n+1}m_{n+1}/(n+1)! - t^n m_n/n!|}{|t^n m_n/n!|} = |t| \cdot L < 1.
\]
By ratio test, we have that \( \sum_{n=0}^{\infty} \frac{t^n m_n}{n!} \) converges for \( |t| < R \), where
\[
R = \begin{cases} 
\frac{1}{L}, & \text{when } 0 < L < \infty, \\
\infty, & \text{when } L = 0.
\end{cases}
\]
Hence the power series \( \sum_{n=0}^{\infty} \frac{t^n m_n}{n!} \) has a positive radius of convergence. Then it follows from Theorem 3.1 that the distribution function \( F \) is moment-determinate.

□

In section 2, we have some results about the absolutely continuous distribution function. Now we consider the case of the discrete distribution function. If the discrete distribution function is concentrated on a finite interval, then the moment problem is determinate. If the discrete distribution function is supported on an infinite interval, then the moment problem remains unknown. So we also want to give some conditions to ensure the uniqueness of the solution of the moment problem. The following result is concerned with the discrete distribution function supported on \( \mathbb{N} \).

**Theorem 3.3** Let \( F = F(x) \) be a discrete distribution function supported on \( \mathbb{N} \) with \( p_k = F(k) - F(k^-) \), \( k \in \mathbb{N} \), \( p_k \geq 0 \) for every \( k \) and \( \sum_{k=1}^{\infty} p_k = 1 \). If
\[
p_k = O(e^{-k^{\alpha}}),
\]
where \( \alpha \geq 1 \), then the distribution function \( F = F(x) \) is moment-determinate.

**Proof.** We show that first
\[
\lim_{n \to \infty} \frac{\left( \sum_{k=1}^{\infty} k^n e^{-k^{\alpha}} \right)^{1/n}}{n} \leq e^{-1} \quad \text{for } \alpha \geq 1. \tag{9}
\]
Let \( f(x) = x^n e^{-x^\alpha} \). Since \( f'(x) = x^{n-1} e^{-x^\alpha} (n - \alpha x^\alpha) \), we have that \( f(x) \uparrow \) for \( x \leq\)
\((\frac{n}{\alpha})^{1/\alpha}\) and \(f(x) \downarrow\) for \(x \geq (\frac{n}{\alpha})^{1/\alpha}\). Since
\[
\sum_{k=1}^{\infty} k^n e^{-k^n} \leq \int_0^{\infty} x^n e^{-x^n} dx + (\frac{n}{\alpha})^{n/\alpha} e^{-n/\alpha}
\]
\[
= \frac{1}{\alpha} \Gamma\left(\frac{n+1}{\alpha}\right) + (\frac{n}{\alpha})^{n/\alpha} e^{-n/\alpha},
\]
we have
\[
\lim_{n \to \infty} \left(\frac{1}{n} \sum_{k=1}^{\infty} k^n e^{-k^n}\right)^{1/n} \leq \lim_{n \to \infty} \frac{\left[\frac{1}{\alpha} \Gamma\left(\frac{n+1}{\alpha}\right) + (\frac{n}{\alpha})^{n/\alpha} e^{-n/\alpha}\right]^{1/n}}{n}
\]
\[
\leq \lim_{n \to \infty} \frac{\left[\Gamma(n+1) + n^{n} e^{-n}\right]^{1/n}}{n} \quad \text{(for } \alpha \geq 1\text{)}
\]
\[
= \lim_{n \to \infty} \frac{(n! + n^{n} e^{-n})^{1/n}}{n}
\]
\[
= \lim_{n \to \infty} \frac{(\sqrt{2\pi} n^{n+1/2} e^{-n} + n^{n} e^{-n})^{1/n}}{n} \quad \text{(by Stirling’s formula)}
\]
\[
= e^{-1}.
\]
Since
\[
\mu_n = \sum_{k=1}^{\infty} k^n p_k \leq \sum_{k=1}^{\infty} k^n \cdot C e^{-k^n},
\]
where \(C\) is a constant, and by (9), therefore,
\[
\lim_{n \to \infty} \frac{\mu_n^{1/n}}{n} \leq \lim_{n \to \infty} \frac{(\sum_{k=1}^{\infty} k^n \cdot C e^{-k^n})^{1/n}}{n}
\]
\[
= \lim_{n \to \infty} \frac{C^{1/n} (\sum_{k=1}^{\infty} k^n e^{-k^n})^{1/n}}{n}
\]
\[
\leq e^{-1} < \infty.
\]
Then it follows from Theorem 2.1 that the distribution function \(F = F(x)\) with \(p_k = O(e^{-k^n})\), where \(\alpha \geq 1\), is moment-determinate.

\[\square\]
References


