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An Investigation of Some Problems Related to
Renewal Process

by

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摘要

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本論文討論關於更新過程(renewal process)的相關問題。更仔細地說，令\(\gamma_t\)代表一更新過程\(A = \{A(t), t \geq 0\}\)的剩餘壽命(residual life)。若\(\text{Var}(\gamma_t) = E^2(\gamma_t) - E(\gamma_t)\)，則當到達間距為離散時，此更新過程為幾何更新過程(geometric renewal process)。另一方面，藉由更新過程隨機和(random sum)的尾部之討論，證明隨機和的\(k\)次方仍滿足新比新的好之性質(new worse than used)。

關鍵詞：指數分佈；幾何分佈；幾何更新過程；舊的比新的好之性質；更新過程；隨機和
Abstract

In this thesis we present some related problems about the renewal processes. More precisely, let $\gamma_t$ be the residual life at time $t$ of the renewal process $A = \{A(t), t \geq 0\}$, $F$ be the common distribution function of the inter-arrival times. Under suitable conditions, we prove that if $\text{Var}(\gamma_t) = E^2(\gamma_t) - E(\gamma_t), \forall t = 0, 1\rho, 2\rho, 3\rho, ...$, then $F$ will be geometrically distributed under the assumption $F$ is discrete. We also discuss the tails of random sums for the renewal process. We prove that the $k$ power of random sum is always new worse than used (NWU).

**Keywords**: exponential distribution; geometric distribution; geometric renewal process; new worse than used; NWU distribution; Renewal process; random sum
1 Introduction

Let \( \{X_n, n \geq 1\} \) be a sequence of independent and identically distributed (i.i.d.) random variables with the common distribution \( F \). Throughout this work we assume \( F(0) = 0 \). Let \( S_0 = 0, S_n = \sum_{i=1}^{n} X_i, n \geq 1 \). Also let \( A \equiv A(t) = \sup\{n \mid S_n \leq t\}, t \geq 0 \), denote the number of renewals in \([0, t]\). The random variables \( \delta_t = t - S_{A(t)} \), \( \gamma_t = S_{A(t)+1} - t \), and \( \beta_t = S_{A(t)+1} - S_{A(t)} \), will be called respectively ”current life at \( t \)”, ”residual life at \( t \)”, and ”total life at \( t \)” of the renewal process \( A \).

Several authors have studied the characterizations of the exponential distribution of \( F \) and hence of the Poisson process by certain properties of the random variable \( \gamma_t \). See, for example, Cinlar and Jagers (1973), Holmes (1974), Gupta and Gupta (1986), Huang and Li (1993), Huang, Li and Su (1993), Huang and Chang (2000).

The results of the above papers are based on the memoryless property of the exponential distribution. On the other hand, the geometric distribution also shares the memoryless property. Some authors such as Huang and Li (1993) also give some corresponding results about characterizing the geometric renewal process when \( F \) is discrete.

Cinlar and Hagers (1973) prove that if \( E(\gamma_t) < \infty \) and \( E(\gamma_t) \) is independent of \( t \) for all \( t \), then the process is Poisson. Gupta and Gupta (1986) extend the above result, namely, under certain conditions, they prove that if \( E(G(\gamma_t)) \) is constant, then the process is also Poisson. Huang, Li and Su (1993) provide a similar result for the arithmetic case. Under some suitable conditions, for a renewal process \( \{A(t), t \geq 0\} \) if \( E(G(\gamma_t)) = c, \forall t = 0, \rho, 2\rho, ... \), they obtain that \( A(t) \) is a geometric renewal process. On the other hand, Huang and Li (1993) also shows that if \( Var(\gamma_t) \) is constant, then \( F \) will be exponential or geometric, for the continuous or discrete case, respectively.
Furthermore, Huang and Chang (2000) prove that if \( \text{Var}(\gamma_t) = E^2(\gamma_t), \forall t \geq 0 \), then \( F \) will be exponentially distributed under the assumption \( F \) is continuous.

Based on the above interesting results, we are motivated to do some investigation. In this work first we give a characterization of the geometric renewal process. More precisely, we prove that if \( \text{Var}(\gamma_t) = E^2(\gamma_t) - E(\gamma_t), \forall t = 0, 1, 2, 3, ... \), then \( F \) will be geometric for the discrete case.

The second part of this thesis is about New Worse than Used (NWU). Later, we will have detail description of NWU property. Assume that \( V \) is a random variable with the probability mass function

\[
Pr\{V = n\} = p_n, n = 0, 1, ...
\]

and \( \{Y_n, n \geq 1\} \) is a sequence of i.i.d. random variables, independent of \( V \) with common distribution function \( H \). Then \( S_V = Y_1 + ... + Y_V (S_0 = 0 \text{ if } V = 0) \) is called the random sum of the random variables \( \{Y_n, n \geq 1\} \). We denote the distribution function of \( S_V \) by \( K \), namely,

\[
K(y) = Pr\{S_V \leq y\} = \sum_{n=0}^{\infty} p_n H^{(n)}(y), y \geq 0,
\]

where \( H^{(n)} \) is the \( n \)-fold convolution of \( H \) with itself, \( H^{(0)}(y) = 1 \text{ if } y \geq 0, \text{ and } 0 \text{ otherwise. The tail of the compound distribution } K \text{ (or of the random sum } S_V \text{) is defined by}

\[
\bar{K}(y) = Pr\{S_V > y\} = 1 - K(y) = \sum_{n=1}^{\infty} p_n \bar{H}^{(n)}(y), y \geq 0,
\]

where \( \bar{H}^{(n)}(y) = 1 - H^{(n)}(y) \). A distribution function \( H \) is NWU if

\[
\bar{H}(y + z) \geq \bar{H}(y)\bar{H}(z),
\]

for \( y \geq 0, z \geq 0 \). Brown (1990) prove that if \( V_0 \) is a geometric random variable, then the geometric sum \( \sum_{n=1}^{V_0} Y_n \) is NWU. His proof has used the theory of stopping
time and the lack of memory property of the geometric distribution. It seems that the result holds only for the geometric sum, since the geometric distribution is the only discrete distribution having the memoryless property. However, later Cai and Kalashnikov (2000) first derive an inequality for the renewal process and an identity in terms of this process for the tail of random sums. By using these results, they prove that a class of random sums holds the NWU property, whatever the distribution $H$ of $\{Y_n, n \geq 1\}$. We will prove that the NWU property holds in the class of $S^k_e$.

Finally, we give some discussion in Section 5.

2 Preliminary

In this section we give some useful theorems and lemmas, which are due to Huang, Li and Su (1993), and Cai and Kalashnikov (2000).

**Theorem 2.1**

Let $\{A(t), t \geq 0\}$ be a renewal process with the inter-arrival time distribution function $F$. Assume $F$ is arithmetic with span $\rho > 0$, and $G$ is a non-decreasing function with $G(0) = 0$. If there exists a finite constant $c > G(\rho)$, such that

$$E(G(\gamma t)) = c, \forall t = 0, \rho, 2\rho, ...,$$

and $c < \sum_{n=0}^{\infty} e^{-\xi n\rho}(G((n + 1)\rho) - G(n\rho)) < \infty$ for some $\xi > 0$, then $\{A(t), t \geq 0\}$ is a geometric renewal process.

Cai and Kalashnikov (2000) first define a class of discrete distributions, which is related to the discrete New Worse than Used (NWU) distributions, and prove that a class of random sums is always NWU.
Lemma 2.1

For any \( y \geq 0 \), the tail \( \bar{K}(y) \) of the random sum \( S_V \) satisfies

\[
\bar{K}(y) = E[a_{A(y)}],
\]

where

\[
a_n = Pr\{V > n\} = \sum_{k=n+1}^{\infty} p_k
\]

is the tail of \( V \).

Lemma 2.2

For any \( t_1 \geq 0 \), \( t_2 \geq 0 \), there exists a random variable \( \hat{A}(t_2) \) such that \( \hat{A}(t_2) \) and \( A(t_1) \) are independent, \( \hat{A}(t_2) \equiv A(t_2) \), and

\[
A(t_1 + t_2) \leq A(t_1) + \hat{A}(t_2) + 1.
\]

Definition 2.1

The discrete distribution \( \{p_n, n \geq 0\} \) of a non-negative integer valued random variable \( V \) is said to be (or to have):

1. discrete decreasing failure rate (D-DFR) if \( a_{n+1}/a_n \) is increasing in \( n \geq 0 \), written as \( V \in D-DFR \), or simply \( V \) is \( D-DFR \);
2. discrete new worse than used (D-NWU) if \( a_{m+n} \geq a_m a_n \), \( m, n = 0, 1, 2, ... \), written as \( V \in D-NWU \), or simply \( V \) is \( D-NWU \).

Example 2.1

If \( V_1 \) be a positive integer valued random variable with the following probability mass function

\[
p_n = Pr\{V_1 = n\} = (1 - q)q^n, \quad n = 0, 1, 2, ..., 0 < q < 1;
\]

thus, \( a_n = \sum_{k=n+1}^{\infty} (1 - q)q^k = q^{n+1} \), so that \( a_{m+n} = q^{m+n+1} > q^{n+1}q^{n+1} = a_m a_n \), and hence \( V_1 \) is \( D-NWU \). Furthermore \( a_{n+1}/a_n = q \) is a constant, which implies that
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\[ V_1 \text{ is } D - DFR. \]

**Definition 2.2**

The discrete distribution \( \{p_n, n \geq 0\} \) of a non-negative integer valued random variable \( V \) is said to be discrete new worse strongly than used (\( DS - NWU \)) if \( a_{m+n+1} \geq a_m a_n, m, n = 0, 1, 2, \ldots \), written as \( V \in DS - NWU \), or simply \( V \) is \( DS - NWU \).

**Example 2.2**

The geometric distribution with
\[ p_n = (1 - q)q^n, n = 0, 1, 2, \ldots, 0 < q < 1, \]
is \( DS - NWU \).

**Theorem 2.2**

If \( V \in DS - NWU \), then the random sum \( \sum_{i=1}^{V} Y_i \) is \( NWU \).

### 3 Characterization related to the geometric characteristic

Let \( \{A(t), t \geq 0\} \) be a renewal process with the inter-arrival time distribution function \( F \). Also assume \( \mu_1 = E(X_1) < \infty \) and \( \mu_2 = E(X_1^2) < \infty \). If \( X_1 \) has a geometric distribution with \( F(x) = 1 - p^x, x = \rho, 2\rho, \ldots \), for some \( 0 < p < 1 \), then \( \gamma_t \) also follows geometric distribution. Hence \( Var(\gamma_t) = E^2(\gamma_t) - E(\gamma_t), \forall t \geq 0 \). For the case that \( F \) is arithmetic we give a characterization of the geometric renewal process.

**Theorem 3.1** Let \( F \) be arithmetic with span \( \rho > 0 \) and \( 0 < F(\rho) < 1 \). Assume
\[ Var(\gamma_t) = E^2(\gamma_t) - E(\gamma_t), \forall t = 0, \rho, 2\rho, 3\rho, \ldots, \]
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then $F$ is geometrically distributed with $F(x) = 1 - p^x$, $x = \rho, 2\rho, \ldots$, for some $0 < p < 1$.

**Proof.** The following arguments are very similar to those of Theorem 2 of Huang and Li (1993). For simplicity we assume $\rho = 1$. First from the assumption we have

$$2E^2(\gamma_t) = E(\gamma_t^2) + E(\gamma_t), \forall t = 0, 1, 2, 3, \ldots$$

(1)

By the usual renewal argument, we obtain

$$E(\gamma_t^2) = \sum_{x=t+1}^{\infty} (x-t)^2 P(X_1 = x) + \sum_{x=1}^{t} E(\gamma_{t-x}^2) P(X_1 = x).$$

Let

$$g(t) = E(\gamma_t).$$

Then (1) implies

$$2g^2(t) = E(\gamma_t^2) + g(t)$$

$$= \sum_{x=t+1}^{\infty} (x-t)^2 P(X_1 = x) + \sum_{x=1}^{t} E(\gamma_{t-x}^2) P(X_1 = x)$$

$$+ \sum_{x=t+1}^{\infty} (x-t)P(X_1 = x) + \sum_{x=1}^{t} E(\gamma_{t-x}) P(X_1 = x)$$

$$= \sum_{x=t+1}^{\infty} (x-t)^2 P(X_1 = x) + 2 \sum_{x=1}^{t} E^2(\gamma_{t-x}) P(X_1 = x)$$

$$+ \sum_{x=t+1}^{\infty} (x-t)P(X_1 = x)$$

$$= \sum_{x=1}^{\infty} x^2 P(X_1 = x) - 2 \sum_{x=1}^{\infty} xtP(X_1 = x) + \sum_{x=1}^{t} t^2 P(X_1 = x)$$

$$+ \sum_{x=1}^{\infty} xP(X_1 = x) - \sum_{x=1}^{t} tP(X_1 = x) - \sum_{x=1}^{t} (x-t)^2 P(X_1 = x)$$

$$- \sum_{x=1}^{t} (x-t)P(X_1 = x) + 2 \sum_{x=1}^{t} E^2(\gamma_{t-x}) P(X_1 = x).$$

Taking the Laplace transforms of both sides, we have

$$2L(g^2(t)) = \frac{\mu_2}{1-e^{-\theta}} - \frac{2\mu_1 e^{-\theta}}{(1-e^{-\theta})^2} + \frac{e^{-\theta}(1+e^{-\theta})}{(1-e^{-\theta})^3}$$
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\[
\begin{align*}
&\quad + \frac{\mu_1}{1-e^{-\theta}} - \frac{e^{-\theta}}{(1-e^{-\theta})^2} - \frac{e^{-\theta}(1+e^{-\theta})}{(1-e^{-\theta})^3} \phi(\theta) \\
&\quad + \frac{e^{-\theta}}{(1-e^{-\theta})^2} \phi(\theta) + 2L(g^2(t))\phi(\theta) \\
&= \frac{\mu_1 + \mu_2}{1-e^{-\theta}} - \frac{2\mu_1 e^{-\theta}}{(1-e^{-\theta})^2} + \frac{2e^{-2\theta}}{(1-e^{-\theta})^3}(1 - \phi(\theta)) \\
&\quad + 2L(g^2(t))\phi(\theta),
\end{align*}
\]

where

\[
\phi(\theta) = E(e^{-\theta X_1}) = \sum_{t=0}^{\infty} e^{-\theta t} P(X_1 = t),
\]

and

\[
L(q(t)) = \sum_{t=0}^{\infty} e^{-\theta t} q(t),
\]

denotes the Laplace transform of the function \(q(t)\) which nevertheless is a function of \(\theta\). It yields

\[
(2L(g^2(t)) - \frac{2e^{-2\theta}}{(1-e^{-\theta})^3})(1 - \phi(\theta)) = \frac{\mu_1 + \mu_2}{1-e^{-\theta}} - \frac{2\mu_1 e^{-\theta}}{(1-e^{-\theta})^2}. \tag{2}
\]

Again

\[
g(t) = E(\gamma_t) = \sum_{x=t+1}^{\infty} (x-t) P(X_1 = x) + \sum_{x=1}^{t} E(\gamma_{t-x}) P(X_1 = x).
\]

We obtain

\[
L(g(t)) = \frac{\mu_1}{1-e^{-\theta}} - \frac{e^{-\theta}}{(1-e^{-\theta})^2} (1 - \phi(\theta)) + L(g(t))\phi(\theta).
\]

It also yields

\[
(L(g(t)) + \frac{e^{-\theta}}{(1-e^{-\theta})^2})(1 - \phi(\theta)) = \frac{\mu_1}{1-e^{-\theta}}. \tag{3}
\]

Comparing (2) and (3), by solving \(1 - \phi(\theta)\) in each equation, we obtain

\[
2L(g^2(t)) = \left(\frac{\mu_1 + \mu_2}{\mu_1}\right)L(g(t)) - \frac{2e^{-\theta}}{1-e^{-\theta}} L(g(t)) \\
\quad + \left(\frac{\mu_1 + \mu_2}{\mu_1}\right) \frac{e^{-\theta}}{(1-e^{-\theta})^2}.
\]
Hence, by the Uniqueness theorem
\[ 2g^2(t) = (1 + \frac{\mu_2}{\mu_1})g(t) - 2 \sum_{x=1}^{t} g(x) + (1 + \frac{\mu_2}{\mu_1})t. \]  
(4)

(4) in turn implies
\[ 2(g^2(t + 1) - g^2(t)) = \left(1 + \frac{\mu_2}{\mu_1}\right)(g(t + 1) - g(t)) - 2g(t) + (1 + \frac{\mu_2}{\mu_1}). \]  
(5)

From (1), we obtain immediately
\[ \mu_2 = 2\mu_1^2 - \mu_1. \]  
(6)

Substituting (6) into (5) yields
\[ 2(g^2(t + 1) - g^2(t)) = 2\mu_1(g(t + 1) - g(t)) - 2g(t) + 2\mu_1, \]
or equivalently
\[ g^2(t + 1) - \mu_1 g(t + 1) = g^2(t) - \mu_1 g(t) - g(t) + \mu_1. \]  
(7)

Since
\[ g(0) = \sum_{x=1}^{\infty} xP(X_1 = x) = E(X_1) = \mu_1, \]
substitute \( t = 0 \) into (7), it yields
\[ g^2(1) - \mu_1 g(1) = \mu_1^2 - \mu_1 \mu_1 - \mu_1 + \mu_1 = 0. \]
Hence \( g(1) = \mu_1 \) or \( g(1) = 0 \), the later contradicts the assumption \( F(1) > 0 \). Therefore, \( g(1) = \mu_1 \).

It can be shown by the induction argument, that \( g(k) = \mu_1, \forall k \geq 0 \). Consequently,
\[ E(\gamma_t) = g(t) = \mu_1, \forall t = 0, 1, 2, 3, ... , \]
is a positive constant. By Theorem 2.1, this in turn implies immediately \( F \) is geometrically distributed.
4 A class of random sums and its NWU property

In this section, we prove that the random sum $(\sum_{i=1}^{V} Y_i)^k$ is NWU, if $V \in DS - NWU$, for any $H$.

**Theorem 4.1** If $V \in DS - NWU$, for every $k \geq 1$, the random sum $(\sum_{i=1}^{V} Y_i)^k$ is NWU.

**Proof.** Let $S_V = Y_1 + \ldots + Y_V$, then

$$T \equiv S_V^k = (\sum_{i=1}^{V} Y_i)^k.$$ 

Also $\bar{T}$ be the tail of $S_V^k$.

By Lemma 2.1 we obtain

$$\bar{T}(y) = Pr\{S_V^k > y\} = Pr\{SV > \sqrt[k]{y}\} = Pr\{V > A(\sqrt[k]{y})\} = E[Pr\{V > A(\sqrt[k]{y}) | A(\sqrt[k]{y})\}] = E[a_A(\sqrt[k]{y})], \quad (8)$$

where

$$a_n = Pr\{V > n\} = \sum_{k=n+1}^{\infty} p_k$$

is the tail of $V$.

By (8) and Lemma 2.2, and since $a_n$ is decreasing in $n \geq 0$, for any $y \geq 0$ and $z \geq 0$, we have

$$\bar{T}(y + z) = E[a_A(\sqrt[k]{y+z})] \geq E[a_A(\sqrt[k]{y}) + a_A(\sqrt[k]{z}) + 1]$$
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\[
\begin{align*}
\geq & \; E[a_A(\sqrt{y})]E[a_{\hat{A}}(\sqrt{\bar{y}})] \\
= & \; E[a_A(\sqrt{y})]E[a_A(\sqrt{z})] \\
= & \; \bar{T}(y)\bar{T}(z),
\end{align*}
\]

where the second inequality follows from the definition of $DS - NWU$, and the second equality follows from the independence of $A(\sqrt{z})$ and $\hat{A}(\sqrt{z})$. The last equality proves that $T$ is $NWU$.

5 Discussion

Let \( \{A(t), t \geq 0\} \) be a renewal process with the inter-arrival time distribution function $F$. Also assume $\mu_1 = E(X_1) < \infty$ and $\mu_2 = E(X_1^2) < \infty$. Let $F$ be an absolutely continuous distribution function with density function $F'$. Assume

\[
E(\gamma_t^2) = aE^2(\gamma_t), \forall t \geq 0, \tag{9}
\]

where $a \neq 2$ is a constant. We would like to know whether condition (9) can characterize $F$.

**Argument.** Again from

\[
E(\gamma_t^2) = aE^2(\gamma_t), \forall t \geq 0,
\]

by the usual renewal argument, we obtain

\[
E(\gamma_t^2) = \int_t^\infty (x - t)^2dF(x) + \int_0^t E(\gamma_{t-x}^2)dF(x).
\]

Letting

\[
g(t) = E(\gamma_t),
\]
then (9) implies

\[ ag^2(t) = E(\gamma_t^2) = \int_t^\infty (x-t)^2 dF(x) + \int_0^t E(\gamma_{t-x}) dF(x) \]

\[ = \int_t^\infty (x-t)^2 dF(x) + a \int_0^t g^2(t-x) dF(x) \]

\[ = \int_0^\infty (x^2 - 2xt + t^2) dF(x) - \int_0^\infty (t-x)^2 dF(x) \]

\[ + a \int_0^t g^2(t-x) dF(x). \]

Taking the Laplace transforms of both sides, we have

\[ aL(g^2(t)) = \mu_2 - \frac{2\mu_1}{\theta} + \frac{2}{\theta^3} \phi(\theta) + aL(g^2(t))\phi(\theta), \tag{10} \]

where

\[ \phi(\theta) = E(e^{-\theta X_1}) = \int_0^\infty e^{-\theta t} dF(x), \]

and

\[ L(q(t)) = \int_0^\infty e^{-\theta t} q(t) dt, \]

denotes the Laplace transform of the function \( q(t) \). From

\[ g(t) = E(\gamma_t) = \int_t^\infty (x-t) dF(x) + \int_0^t E(\gamma_{t-x}) dF(x), \]

we obtain

\[ L(g(t)) = \frac{\mu_1}{\theta} - \frac{1}{\theta^2} + \frac{1}{\theta^3} \phi(\theta) + L(g(t))\phi(\theta). \tag{11} \]

Now (10) and (11) can be rewritten as

\[ (aL(g^2(t)) - \frac{2}{\theta^3})(1 - \phi(\theta)) = \frac{\mu_2}{\theta} - \frac{2\mu_1}{\theta^2} \tag{12} \]

and

\[ (L(g(t)) + \frac{1}{\theta^2})(1 - \phi(\theta)) = \frac{\mu_1}{\theta}. \tag{13} \]
Comparing (12) and (13), by solving $1 - \phi(\theta)$ in each equation, we obtain

$$aL(g^2(t)) = \frac{\mu_2}{\mu_1} L(g(t)) + \frac{\mu_2}{\mu_1} \frac{1}{\theta^2} - \frac{2}{\theta} L(g(t)), \forall t \geq 0. \tag{14}$$

Hence, by the Uniqueness theorem, (14) implies

$$ag^2(t) = \frac{\mu_2}{\mu_1} g(t) + \frac{\mu_2}{\mu_1} t - 2 \int_0^t g(x) dx, \forall t \geq 0. \tag{15}$$

The assumption that $F$ is differentiable implies $g$ is also differentiable. Hence, taking the derivatives of both sides of (15) with respect to $t$, it follows

$$\frac{d}{dt}(2ag(t) - \frac{\mu_2}{\mu_1}) g'(t) = \frac{\mu_2}{\mu_1} - 2g(t), \forall t \geq 0.$$

Let

$$U(g) = g'(t) = \frac{\frac{\mu_2}{\mu_1} - 2g(t)}{2ag(t) - \frac{\mu_2}{\mu_1}}.$$

and from (9) we know

$$g(0) = E(\gamma_0) = \frac{E(\gamma_0^2)}{aE(\gamma_0)} = \frac{\int_0^\infty x^2 dF(x)}{\int_0^\infty xdF(x)} = \frac{E(X_1^2)}{aE(X_1)} = \frac{\mu_2}{a\mu_1}.$$

The system is

$$\begin{cases} g'(t) = \frac{\frac{\mu_2}{\mu_1} - 2g(t)}{2ag(t) - \frac{\mu_2}{\mu_1}} = U(g), \\ g(0) = \frac{\mu_2}{a\mu_1}. \end{cases} \tag{*}$$

Since $U(g)$ is differentiable, and

$$U''(g) = \frac{-2(2ag(t) - \frac{\mu_2}{\mu_1}) - 2a(\frac{\mu_2}{\mu_1} - 2g(t))}{(2ag(t) - \frac{\mu_2}{\mu_1})^2}$$

is continuous. Then $U(g)$ is continuously differentiable on $(\frac{\mu_2}{2a\mu_1}, \infty)$ that contains initial point $g(0) = \frac{\mu_2}{a\mu_1}$. And if $U(g)$ is continuously differentiable on $(\frac{\mu_2}{2a\mu_1}, \infty)$ that contains initial point $g(0) = \frac{\mu_2}{a\mu_1}$, then (*) has a unique solution from Boyce and DiPrima (1992).
If \( U(g) = 0 \), then \( g = \frac{\mu_2}{2\mu_1} \), which is an equilibrium point by Boyce and DiPrima (1992). If \( a \neq 2 \), then \( g = \frac{\mu_2}{a\mu_1} \) is not an equilibrium point. Thus \( g(t) \) is not a constant.

Case 1. \( a = 1 \)

If \( a = 1 \), then the system (*) is

\[
\begin{align*}
g'(t) &= \frac{\mu_2}{2\mu_1} - 2g(t) - \frac{\mu_2}{\mu_1}, \\
g(0) &= \frac{\mu_2}{\mu_1}.
\end{align*}
\]

Solving it, we obtain

\[ g(t) = -t + \frac{\mu_2}{\mu_1}. \]

If \( t > \frac{\mu_2}{\mu_1} \), then \( g(t) = E(\gamma_t) < 0 \), a contradiction, because the expected value of \( \gamma_t \) cannot be negative.

Case 2. \( 1 < a < 2 \)

If \( 1 < a < 2 \), then the system (*) is

\[
\begin{align*}
g'(t) &= \frac{\mu_2}{2a\mu_1} - 2g(t) - \frac{\mu_2}{\mu_1} = U(g), \\
g(0) &= \frac{\mu_2}{a\mu_1},
\end{align*}
\]

and \( g'(0) < 0 \). If \( 1 < a < 2 \), then \( \frac{\mu_2}{2a\mu_1} > \frac{\mu_2}{2\mu_1} \) and \( H(\frac{\mu_2}{2\mu_1}) = 0 \), so \( \frac{\mu_2}{2\mu_1} \) is still equilibrium point. If the initial point is \( \frac{\mu_2}{a\mu_1} \), then the solution of (*) is \( g(t) \). Since \( g'(0) < 0 \), there exists a \( t_0 > 0 \) such that \( \frac{\mu_2}{2a\mu_1} < g(t_0) < \frac{\mu_2}{a\mu_1} \). And if there exists a \( t_1 > 0 \) such that \( g(t_1) \leq \frac{\mu_2}{2\mu_1} \), by Intermediate-Value Theorem (see Fitzpatrick (1996)), there exists a \( t'_1 > 0 \) such that \( g(t'_1) = \frac{\mu_2}{2\mu_1} \), which also is a contradiction, because no solutions can enter a equilibrium point in a finite time (see Boyce and DiPrima (1992)). Thus we detect that \( g(t) \) locate within \( \frac{\mu_2}{2\mu_1} \) and \( \frac{\mu_2}{a\mu_1} \), the slope of \( g(t) \) is negative. We know \( g(t) \) run downside and approach to the equilibrium point \( \frac{\mu_2}{2\mu_1} \), but there is no intersection.
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point with the equilibrium point.

Case 3. \( a > 2 \)

If \( a > 2 \), then the system (*) is

\[
\begin{align*}
g'(t) &= \frac{\mu_2}{\mu_1} - 2g(t) + \frac{\mu_2}{\mu_1} \cdot a g(t) - \frac{2\mu_2}{\mu_1}, \\
g(0) &= \frac{\mu_2}{\mu_1},
\end{align*}
\]

and \( g'(0) > 0 \). Similarly to Case 2, we also respect that \( g(t) \) locates within \( \frac{\mu_2}{2\mu_1} \) and \( \frac{\mu_2}{\mu_1} \). Hence the slope of \( g(t) \) is positive. We know \( g(t) \) run upside and approach to the equilibrium point \( \frac{\mu_2}{2\mu_1} \), but there is no intersection point with the equilibrium point.
References


